EFFECTIVE ELASTIC MODULI OF THIRD-ORDER COMPOSITE MATERIALS

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Certain problems of the structural mechanics of composite materials cannot be solved in the framework of a linear theory (for example, problems of stability and wave propagation in prestrained inhomogeneous materials). The present paper proposes a method for calculation of macroscopic elastic moduli of the second and third orders. A microinhomogeneous medium is investigated in the approximation of a geometrically linear theory. Estimates of the moments of strain fields in the components are obtained by using a nonlinear formulation of the effective field method [1-4]. The method rests on a solution of the problem of binary interaction of inclusions present in the effective field. The deformations within each inclusion are assumed to be homogeneous. The second moments of the strain fields in the components are assumed to be uniform.

1. General Relations. In a macrovolume w with the characteristic function W, we consider a mixture of elastic components whose mechanical properties are described by a geometrically linear theory (under the classification of [5], it is the second variant of small initial deformations). The strain tensor ε_{ij} is linked with the components of the displacement vector u_i by the relation

$$\varepsilon_{ij} = (u_{i,j}+u_{j,i})/2,$$

The characteristic equation appears as

$$\sigma = L\varepsilon + \mathscr{L}\,\varepsilon \otimes \varepsilon. \tag{1.1}$$

In particular, for the Murnaghan potential

$$\Phi = (1/2)\,\lambda A_1^2 + \mu A_2\,(a/3)\,A_1^3 + bA_1A_2 + (c/3)\,A_3 \tag{1.2}$$

($A_1 = \varepsilon_{ij}$, $A_2 = \varepsilon_{ij}\varepsilon_{ij}$, $A_3 = \varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki}$ are the algebraic invariants of the strain tensors). We obtain from (1.1) and (1.2) and the relation $\sigma_{ij} = (1/2)(\partial/\partial\varepsilon_{ij} + \partial/\partial\varepsilon_{ji})\Phi$ the following expressions:

$$L_{ijkl} = 3kN^1_{ijkl} + 2\mu N^2_{ijkl},\ N^1_{ijkl} = (1/3)\,\delta_{ij}\delta_{kl},$$
$$N^2_{ijkl} = I_{ijkl} - N^1_{ijkl},\ I_{ijmn} = (\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})/2,$$
$$\mathscr{L}_{ijklmn} = 3a\delta_{ij}N^1_{mnkl} + b\,(\delta_{ij}I_{mnkl} + \delta_{mn}I_{ijkl} + \delta_{kl}I_{mnij}) + cJ_{ijmnkl},$$
$$J_{ijmnkl} = (I_{ipkl}I_{pjmn} + I_{ipmn}I_{pjkl})/2.$$

A matrix with a characteristic function V_0 and the moduli $L_0, \mathscr{L}_0$ contains the set $X = (V_k, L^{(k)}, \mathscr{L}^{(k)})$ of ellipsoids v_k with characteristic functions V_k, the half-axes a^i_k, the orientations ω_k, the centers x_k, and the moduli $L^{(k)}$, $\mathscr{L}^{(k)}$.

Here and in what follows, we use the notations of tensor equations, omitting indices. The product of tensors is assumed to be their convolution by inner indices. The direct tensor product is denoted by the symbol $\otimes$. Standard hypotheses for microinhomogeneous media are adopted [1-6]: All random fields are statistically homogeneous and ergodic. Thus, the statistical averaging over an ensemble can be replaced by averaging over a characteristic volume:

$$\langle(\cdot)\rangle = (\mathrm{mes}\ w)^{-1}\int(\cdot)W(r)dr,\ \langle(\cdot)\rangle_\alpha = (\mathrm{mes}\ v_\alpha)^{-1}\int(\cdot)V_\alpha(r)dr$$
$$(\alpha = 0, 1, \ldots).$$

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We also use the notation $\langle(\cdot)|x_2; x_1\rangle$ for the conditional average over an ensemble X under the condition that there are inclusions at the points x_1 and x_2, and $x_1 \neq x_2$. The components refer to different phases X_α if at least one of the parameters a_α, ω_α, $L^{(\alpha)}$, $\mathscr{L}^{(\alpha)}$ has different values.

The equilibrium equation for a microinhomogeneous medium, disregarding mass forces, appears as

$$\nabla[(L_0 + L_1(x))\varepsilon(x) + (\mathscr{L}_0 + \mathscr{L}_1(x))\varepsilon(x) \otimes \varepsilon(x)] = 0, \tag{1.3}$$

where ∇ is the operation of symmetrized gradients:

$$L_1(x) = \sum_k V_k(x)\left(L^{(k)} - L_0\right), \ \mathscr{L}_1 = \sum_k V_k(x)\left(\mathscr{L}^{(k)} - \mathscr{L}_0\right).$$

Equation (1.3) is nonlinear. For obtaining final results that can be visualized, we adopt linearization of (1.3), which assumes the homogeneity of $\varepsilon(x) \otimes \varepsilon(x)$ within the phase X_α: $\varepsilon(x) \otimes \varepsilon(x) \equiv \langle\varepsilon(x) \otimes \varepsilon(x)\rangle_\alpha$ at $x \in X_\alpha$. We denote $q(x) = (L_0 + L_1{}^{(\alpha)})^{-1}(\mathscr{L}_0 + \mathscr{L}_1^{(\alpha)})\langle\varepsilon(x) \otimes \varepsilon(x)\rangle_\alpha$, $q(x) = q_0$ at $x \in X_0$, $q_1(x) = \sum\limits_{k=1} (q(x) - q_0)V_k$. The rule of calculation of the piecewise constant tensor of the second rank q is described below. In our notations, (1.3) appears as

$$\nabla(L_0 + L_1(x))[\varepsilon(x) + q(x)] = 0 \tag{1.4}$$

Within notations, it coincides with its counterpart relation from linear theories of gas-saturated porous media [2] and thermal elasticity [3] of microinhomogeneous media. Thus, we can employ for solution of (1.4) the techniques of the effective field method proposed earlier [1-3]. Specifically, with the aid of the fundamental solution G of the equation for the equilibrium of a homogeneous linearly elastic medium with the modulus L_0, we reduce (1.4) to an integral equation for modified deformation $e = \varepsilon - q_0$:

$$e(x) = \langle e\rangle + \int \nabla\nabla G(x-y)\{L_1(y)e(y) - (L_0 + L_1(y))q_1(y) - [\langle L_1 e\rangle - \langle(L_0 + L_1)q_1\rangle]dy. \tag{1.5}$$

Expressing (1.5) in stresses and taking into account that $\langle\sigma\rangle = \sigma^0$, we obtain

$$\sigma(x) = \sigma^0 + \int \Gamma(x-y)\{M_1(y)\sigma(y) - q_1(y) - [\langle M_1\sigma\rangle - \langle q_1\rangle]\}dy.$$

Here, $M_0 = L_0{}^{-1}$; $M_0 + M_1(x) \equiv M_0 + M_1{}^{(k)} \equiv (L_0 - L_1{}^{(k)})^{-1}$ at $x \in v_k$ is the compliance of the k-th inclusion; $\Gamma(x - y) = -L_0(I\delta(x - y) + \nabla\nabla G(x - y)L_0)$; δ is the δ-function.

The effective moduli of the second and third orders in the relation

$$\langle\sigma\rangle = L_* \langle\varepsilon\rangle + \mathscr{L}_* \langle\varepsilon\rangle \otimes \langle\varepsilon\rangle \tag{1.6}$$

can be found by averaging local equation (1.1):

$$L_* = L_0 + \langle L_1 A^*\rangle, \ \mathscr{L}_* = \sum_{\nu=1} \langle L_1^{(\nu)}\mathscr{F}_1^{(\nu)}\rangle_\nu + \sum_{\alpha=0} \xi_\alpha \mathscr{L}^{(\alpha)}\mathscr{F}_2^{(\alpha)}, \tag{1.7}$$

where $\xi_\alpha = \langle V_\alpha\rangle$; the tensors of the fourth rank A^*, the sixth rank $\mathscr{F}_1$, and the eighth rank $\mathscr{F}_2$ define the average concentration of deformations in a component $X_\alpha \ni x$

$$\langle\varepsilon\rangle_\alpha = A_\alpha^* \langle\varepsilon\rangle + \mathscr{F}_1(x)\langle\varepsilon\rangle \otimes \langle\varepsilon\rangle, \ \langle\varepsilon \otimes \varepsilon\rangle_\alpha = \mathscr{F}_2(x)\langle\varepsilon\rangle \otimes \langle\varepsilon\rangle. \tag{1.8}$$

2. Evaluation of Average Deformations in the Components. We fix an arbitrary realization of the field X and examine an effective field $\bar{e}(x)$, $x \in v_k$, which contains an inclusion

$$\bar{e}(x) = \langle e\rangle + \int U(x-y)\{V(y;x)[L_1(y)e(y) + (L_0 + L_1(y))q_1(y)] - [\langle L_1 e\rangle + \langle(L_0 + L_1)q_1\rangle]\}dy \tag{2.1}$$

$$\left(V(y;x) = V(y) - V_k(x), \ V(y) = \sum_{k=1} V_k(y), \ U = \nabla\nabla G\right).$$

The field X, and therefore also $\bar{e}$, are random. In order to determine $\langle\bar{e}\rangle$, we will make use of the hypothesis of the effective field method described in detail in [1, 2]: 1) the field of $\bar{e}$ is homogeneous in the neighborhood of each point inclusion; 2) every n (n > 1) inclusions exist in a generally inhomogenenous field $\bar{e}_{1,\ldots,n}$ of their own.

From the homogeneous field $\bar{e}$ we can determined unequivocally a homogeneous field of strains inside each inclusion [2]:

$$e(x) = A_k\left(\bar{e} - P_k\left(L_0 + L_1^{(k)}\right)q_1\right), \quad A_k = \left(I + P_k L_1^{(k)}\right)^{-1}, \tag{2.2}$$

where $x \in v_k$ and the constant tensor $P_k = -\int U(x-y)V_k(y)dy$ $(x \in v_k)$ is known.

We will describe the structure of a composite material by a function $\varphi(v_m/v_k)$ – the conditional density of the distribution of the m-th inclusion in the region v_m at a fixed inclusion in the region v_m at a fixed inclusion in the region v_k. Since inclusions do not overlap, we assume that

$$\varphi(v_m \mid v_k) \equiv \psi(\omega_m)\left(1 - V'_{km}\right) f_{km}(|r|)(\mathrm{mes}\, W)^{-1}. \tag{2.3}$$

From the normalization condition $\langle\psi(\omega_m)\rangle = 1$ in the absence of the near oder $f_{km}(|r|) = n_\nu$, $\nu = 1, 2, \ldots$, if $v_m \in X_\nu$; n_ν a countable concentration of inclusions of the components X_ν, is linked with the volumetric concentration $\xi_\nu = (4/3)\pi\, a_\nu^1 a_\nu^2 a_\nu^3\, n_\nu$; V_{km}' is the characteristic function of a sphere v_{km}' with the center x_k and the radius $a_{km} = \min a_m^i + \max a_k^i$.

Averaging (2.1) on the set $X(\cdot|x_k)$, by means of (2.3) and assuming hypothesis 1 of the effective field, we obtain

$$\langle \bar{e}_k\rangle = \langle e\rangle + \int U(x-y)\{\langle [L_1 A(y)\bar{e}(y) + A(y)(L_0 + L_1)q_1]V(y;x)|y;x\rangle - [\langle L_1 A\bar{e}\rangle + \langle A(L_0 + L_1)q_1\rangle]\}dy. \tag{2.4}$$

For calculating conditional moments in (2.4), we adopt hypothesis 2 with n = 2 and the first approximations of the solution of the problem of binary interaction of inclusions in a homogeneous matrix [3]. By analogy with [3], we write

$$\begin{gathered} \bar{e} = D(\langle e\rangle + \langle F\rangle), \\ D = \left(I - P_0\langle R\rangle - \int \langle\!\langle J_{12}\left(1 - V'_{12}\right) f_{12}\rangle\!\rangle_{12}\, dx_2\right)^{-1}, \\ F = P_0 R q_1 + \int \langle\!\langle T_{12}\left(1 - V'_{12}\right) f_{12} q_1\rangle\!\rangle_{12}\, dx_2, \end{gathered} \tag{2.5}$$

where $R_k = L_1^{(k)} A_k \bar{v}_k$; $\bar{v}_k = \mathrm{mes}\, v_k$; $P_0 = P(v_{km}')$; $J_{12} = UR_2UR_1$; $T_{12} = UR_2UA_1(L_0 + L_1^{(k)})$; $\langle\!\langle\cdot\rangle\!\rangle_{km}$ denote the operation of averaging with respect to ω_k, ω_m, a_{km} and the positions x_m of the sphere of radius $|r| = |x_k - x_m|$ with center at x_m.

From (2.5) we determine the mean strain in the components of the inclusions X_ν (ν = 1, 2, 3, ...) and the matrix X_0:

$$\begin{gathered} \langle\varepsilon_\nu\rangle = A_\nu D\{\langle\varepsilon\rangle - P_\nu(L_0 + L_1)D^{-1}q_1 + q_0 + \langle F\rangle\}, \\ (1 - \xi)\langle\varepsilon\rangle_0 = \langle\varepsilon\rangle - \sum_{\nu=1} \xi_\nu\langle\varepsilon\rangle_\nu, \quad A_\nu^* = A_\nu D, \\ A_0^* = (I - \langle ADV\rangle)(1 - \xi)^{-1}, \quad \xi = \langle V\rangle. \end{gathered} \tag{2.6}$$

Expressions of A^* in (2.6) make it possible to determine the effective second-order modulus L_* from (1.7). Assuming equiprobable orientation of inclusions, the tensors $\langle R\rangle$, $\langle\!\langle J_{12}\rangle\!\rangle_{12}$, $\langle\!\langle T_{12}\rangle\!\rangle_{12}$, D, L_* are isotropic, and

$$\begin{gathered} \langle\!\langle J_{12}\rangle\!\rangle_{12} = \left(3J_{12}^1, 2J_{12}^2\right), \quad \langle\!\langle T_{12}\rangle\!\rangle_{12} = \left(3T_{12}^1, 2T_{12}^2\right), \quad 3J_{12}^1 = 2\beta^2(3\bar{k}_1)(2\bar{\mu}_2)|r|^{-6}, \\ 2J_{12}^1 = (2/5)[\beta^2(3\bar{k}_2)(2\bar{\mu}_1) + (2\bar{\mu}_1)(2\bar{\mu}_2)(7\gamma^2 - \eta^2/4 + 2\beta\eta)]|r|^{-6}, \\ \beta = (3k_0 + 4\mu_0)^{-1}, \eta = (3\mu_0)^{-1}, \gamma = (3k_0 + 4\mu_0)[3\mu_0(3k_0 + 4\mu_0)]^{-1}, \end{gathered}$$

where for the isotropic tensor $B_{ijk\ell}$ we adopt the notations

$$B = (3B^1, 2B^2) = 3B^1N^1 + 2B^2N^2,$$

$$\langle L_1^{(i)} A_i\rangle \prod_{j=1}^{3} a_i^j = \left(3\bar{k}_i, 2\bar{\mu}_i\right), \quad \langle L_1 A\rangle = \int L_1 A\psi(\omega)\, d\omega.$$

To obtain $3T_{12}^1$, $2T_{12}^2$ we must in $2J_{12}^2$ replace $(3\bar{k}_1, 2\bar{\mu}_1)$ with $(3t_1, 2t_2) = \langle(L_0 + L_1^{(1)})A_1\rangle \prod_{j=1}^{3} a_i^j$.

3. Calculation of $\mathscr{F}_1, \mathscr{F}_2, \mathscr{L}_*$. So far we have assumed that q_0 and q_1 are known. However, by assumption, these tensors are dependent on the second moments of strain fields in the components. In that case, problem (1.3) is nonlinear. For estimating constant tensors q_α ($\alpha = 0, 1, \ldots$) we employ the method of successive approximations from [4]: $q_\alpha^{(n+1)} = \langle\varepsilon^{(n)} \otimes \varepsilon^{(n)}\rangle_\alpha (L_0 + L_1^{(\alpha)}(x))^{-1}(\mathscr{L}_0 + \mathscr{L}_1^{(\alpha)} x))$, $q_\alpha^{(0)} = 0$. The values of $\langle\varepsilon^{(n)} \otimes \varepsilon^{(n)}\rangle$ are estimated by using the method of [1] for known $q_\alpha^{(n)}$. For reducing derivations in (1.8) we take the first iterative approximation of $\langle\varepsilon^{(0)} \otimes \varepsilon^{(0)}\rangle_\alpha$ and $\langle\varepsilon^{(1)}\rangle_1$. The calculation of the second moment of $\langle\varepsilon^{(0)} \otimes \varepsilon^{(0)}\rangle_\alpha$ can be conducted by constructing a correlation function of strain fields by using the method of [1]. If in that case the solution of the problem of binary interaction of inclusions [1] by successive approximations takes into account, as in (2.5), the terms of the series that decrease at infinity not faster than J_{12}, it can be demonstrated that

$$\langle\varepsilon^{(0)} \otimes \varepsilon^{(0)}\rangle_\alpha = \langle\varepsilon^{(0)}\rangle_\alpha \otimes \langle\varepsilon^{(0)}\rangle_\alpha \ (\alpha = 0, 1, \ldots). \tag{3.1}$$

Comparing (1.8) with (2.6) and (3.1), we obtain

$$\mathscr{F}_2^{(\alpha)} = A_\alpha^* \otimes A_\alpha^*, \quad \mathscr{F}_1^{(\nu)} = A_\nu D\{L_0^{-1}\mathscr{L}_0\mathscr{F}_2^{(0)} - P_\nu D^{-1}(\mathscr{L}^{(\nu)}\mathscr{F}_2^{(\nu)} -$$
$$- L^{(\nu)}L_0\mathscr{L}_0\mathscr{F}_2^{(0)}) + P_0\langle[(\mathscr{L}_0 + \mathscr{L}_1)A\mathscr{F}_2 - (L_0 + L_1)AL_0^{-1}\mathscr{L}_0\mathscr{F}_2^{(0)}]V\rangle +$$
$$+ \int \langle\!\langle UR_2U[\mathscr{L}^{(1)}A_1\mathscr{F}_2 - LA_1L_0^{-1}\mathscr{L}_0\mathscr{F}_2^{(0)}](1 - V_{12}')f_{12}\rangle\!\rangle_{12}\, dx_2\} - L_0^{-1}\mathscr{L}_0\mathscr{F}_2^{(0)} \quad (\alpha = 0, 1, \ldots;\ \nu = 1, 2, \ldots).$$

Likewise, we define $\mathscr{F}_1^{(0)}$. Substituting the values of $\mathscr{F}_1^{(\alpha)}, \mathscr{F}_2^{(\alpha)}$ from (3.1) into (1.7), we find the effective elastic modulus of the third order:

$$\mathscr{L}_* = \sum_{\alpha=0} \xi_\alpha \mathscr{L}^{(\alpha)} \otimes A_\alpha^* \otimes A_\alpha^* \otimes A_\alpha^*. \tag{3.2}$$

This expression is a generalization extending the corresponding relation from [6] to an arbitrary number of components. For two-component composite materials, (3.2) coincides within notations with the expression in [6]. The sole difference is in specific equations for A_α^*, i.e., in the solution of the linearly elastic problem.

Generally, the tensors A_α^*, and therefore also L_*, $\mathscr{L}_*$ are anisotropic. At an equiprobable orientation of the inclusions A_α^*, L_*, $\mathscr{L}_*$ are isotropic: $A_\alpha^* = (3r_\alpha, 2s_\alpha)$,

$$\mathscr{L}_{*ijmnkl} = a_*\delta_{ij}\delta_{mn}\delta_{kl} + b_*(\delta_{ij}I_{mnkl} + \delta_{mn}I_{ijkl} + \delta_{kl}I_{ijkl}) + c_*J_{ijmnkl},$$
$$a_* = \sum_{\alpha=0} 3\xi_\alpha [9a_\alpha r_\alpha^3 + 3b_\alpha p_\alpha r_\alpha(3r_\alpha + 2s_\alpha) + c_\alpha p_\alpha^2(p_\alpha + 2s_\alpha)],$$
$$b_* = \sum_{\alpha=0} \xi_\alpha (2s_\alpha)^2(3b_\alpha r_\alpha + c_\alpha p_\alpha), \quad c_* = \sum_{\alpha=0} \xi_\alpha c_\alpha (2s_\alpha)^3$$

($3p_\alpha = 3r_\alpha - 2s_\alpha$; a_α, b_α, c_α are the components of $\mathscr{L}^{(\alpha)}$).

Example. Since differences in the estimates of $\mathscr{L}_*$ on the basis of our method and the results of [6] are connected with the solution of the linearly elastic problem of calculation of A_α^*, we will make a quantitative comparison of A_α^* computed by the method of condi-

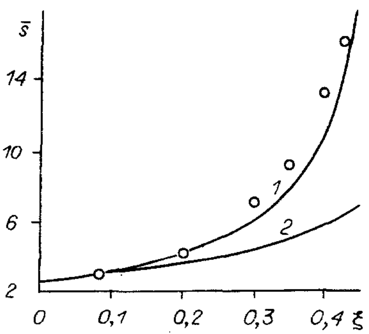

Fig. 1

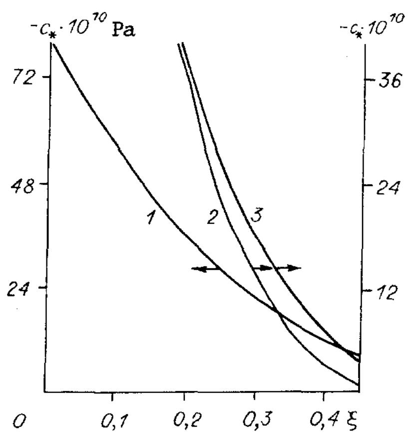

Fig. 2

tional moments of [6] and by the effective field method of [1]. For rigid spherical inclusions of the same size in an incompressible matrix we obtain from (2.2), (2.6), and (1.8)

$$2\bar{s} = 2s_1(\mu_*/\mu_0 - 1) = 5(2 - 31\xi/8)^{-1} \text{ and } 2\bar{s} = (5/2 - \xi)(1 - \xi)^{-2}$$

by the method of conditional moments {curves 1 and 2 in Fig. 1, respectively; the points represent the experimental data of [7] on the variations of the effective Newtonian viscosity of suspensions in response to a growth in ξ, replotted in the coordinates $\bar{s} \sim \xi$ with the aid of (1.7)}. For spherical and flat spheroidal pores similar estimates have been compared in [2]. Figure 2 plots $c_*(\xi)$ calculated from (2.6) and (3.2) for 09G2S steel with spherical pores of the same size and the following parameters (Pa): $\lambda_0 = 9.44 \cdot 10^{10}$, $\mu_0 = 7.9 \cdot 10^{10}$, $a_0 = -82.5 \cdot 10^{10}$, $b_0 = -30.9 \cdot 10^{10}$, $c_0 = -79.9 \cdot 10^{10}$. The value of $c_*(\xi)$ at $\xi = 0.4$ on curve 1 in Fig. 2 is greater by 20% than the estimate by the method of conditional moments [6]. We should note that for a porous medium the ratio of c_* values based on (2.6) and (3.2) to those calculated by the method of [6] is equal to the cube of s_0. Therefore, the difference in estimates of c_* by (2.6) and (3.2) and by [6] will grow as k increases and as the shape of the inclusions approaches a spheroid [2]. Indeed, for spherical pores and $k_0 = \infty$, we show in Fig. 2 the values of $c_*/[c_0(1 - \xi)] \sim \xi$ (curves 2 and 3) calculated from formulas of [6] and from (2.6) and (3.2), respectively.

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